

Name: Solutions.

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1. Fix $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$

(a) How many vectors are in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Is \vec{w} in this set?

3 vectors. \vec{w} is NOT in the set.

(b) How many vectors are in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

∞ -many vectors.

(c) Is \vec{w} in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? You must justify your answer.

does

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{w} \quad \text{have a soln?}$$

\Leftrightarrow

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 0 & 5 & 4 \\ 1 & -2 & 5 & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & -4 & 2 & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -4 & -2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

echelon form
units (00010)
 \Rightarrow has soln

(d) If \vec{w} is in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, find coordinates for \vec{w} relative to $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Use these coordinates to set up a vector equation, and double check that $\vec{w} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\begin{cases} x_1 & = -1 \\ x_2 & = -1 \\ x_3 & = 1 \end{cases}$$

check:

$$-1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + -1 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix} \quad \checkmark$$

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2. Do the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ -6 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}$ span \mathbb{R}^3 ?

is $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{b}$ consistent for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

$$\left[\begin{array}{ccc|c} 2 & -4 & 0 & b_1 \\ 2 & -6 & -2 & b_2 \\ -2 & 0 & -4 & b_3 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} -2 & -4 & 0 & b_1 \\ 0 & -2 & -2 & b_2 - b_1 \\ 0 & -4 & -4 & b_3 + b_1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -4 & 0 & b_1 \\ 0 & -2 & -2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2(b_2 - 2b_1) \end{array} \right]$$

the matrix can contain $[0 \dots 0 | 0]$ for some \vec{b}

\Rightarrow
NOT always consistent.

3. Find all possible h so that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} h \\ 2 \end{bmatrix}$ span \mathbb{R}^2 .

find h s.t.

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} h \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & h & b_1 \\ 5 & 2 & b_2 \end{array} \right] \text{ is always consistent}$$

$$\sim \left[\begin{array}{cc|c} 1 & h & b_1 \\ 0 & 2-5h & b_2-5b_1 \end{array} \right]$$

cannot contain $[0 \dots 0 | \blacksquare]$ when \Leftrightarrow

$$2-5h \neq 0$$

$$2 \neq 5h$$

$$h \neq \frac{2}{5}$$

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4. Compute the following matrix products, showing all work, or state that they are undefined

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ DNE. you need # rows in X
columns in A

(b) $\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 2 & 4 \\ 2 & 2 \\ 2 & 0 \end{bmatrix}$. Determine if $\vec{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ is a linear combination of the columns of the matrix

A. If it is a linear combination, find coordinates \vec{b} with respect to the columns of A . Check your answer using a vector or matrix equation.

Does $\begin{bmatrix} 2 & 4 \\ 2 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ have a soln?

$\left[\begin{array}{cc|c} 2 & 4 & 3 \\ 2 & 2 & 5 \\ 2 & 0 & 7 \end{array} \right] \sim \left[\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{array} \right] \sim \left[\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{array} \right]$

By thm 2, $[0 \ 0 \ 0]$ doesn't occur
 \Rightarrow has a soln.

$\sim \left[\begin{array}{cc|c} 2 & 0 & 7 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{array} \right]$

unique soln

$\begin{cases} x_1 = 7/2 \\ x_2 = -1 \end{cases}$

Check

$\frac{7}{2} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \checkmark$

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6. Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$. Does the system $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^2$?

You must justify your answer.

$$\left[\begin{array}{cc|c} 2 & 5 & * \\ 1 & 4 & * \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 4 & * \\ 2 & 5 & * \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 4 & * \\ 0 & -3 & * \end{array} \right]$$

NOTE

Cannot contain

$$[0 \ 0 \ | \ \blacksquare],$$

regardless of \vec{b} .

it DOES have a soln for every $\vec{b} \in \mathbb{R}^2$.

7. Let $A = \begin{bmatrix} 2 & -2 & 6 & 5 \\ 0 & -2 & 2 & 2 \\ 4 & -2 & 10 & 8 \end{bmatrix}$.

Determine if the system $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^3$.

$$\left[\begin{array}{cccc|c} 2 & -2 & 6 & 5 & * \\ 0 & -2 & 2 & 2 & * \\ 4 & -2 & 10 & 8 & * \end{array} \right] \sim \left[\begin{array}{cccc|c} 2 & -2 & 6 & 5 & * \\ 0 & -2 & 2 & 2 & * \\ 0 & 2 & -2 & -2 & * \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 2 & -2 & 6 & 5 & * \\ 0 & -2 & 2 & 2 & * \\ 0 & 0 & 0 & 0 & * \end{array} \right]$$

Can contain $[0 \dots 0 \ | \ \blacksquare]$
for some \vec{b}

It does NOT have a soln for some \vec{b} in \mathbb{R}^3

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8. Do the columns of the matrix $A = \begin{bmatrix} 10 & 14 \\ 5 & 7 \end{bmatrix}$ span \mathbb{R}^2 ?

You must justify your answer with an argument.

Columns of A span $\mathbb{R}^2 \Leftrightarrow A\vec{x} = \vec{b}$ has solution
for every $\vec{b} \in \mathbb{R}^2$

$$\left[\begin{array}{cc|c} 10 & 14 & * \\ 5 & 7 & * \end{array} \right] \sim \left[\begin{array}{cc|c} 5 & 7 & * \\ 5 & 7 & * \end{array} \right] \sim \left[\begin{array}{cc|c} 5 & 7 & * \\ 0 & 0 & * \end{array} \right]$$

\uparrow $[0 \ 0 \ | \ *]$ can occur
for some \vec{b}

\Rightarrow NOT every system has a sol.

So columns of A do NOT span \mathbb{R}^2

9. Do the columns of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 0 & 2 & 5 \end{bmatrix}$ span \mathbb{R}^3 ?

You must justify your answer with an argument.

Columns of A span $\mathbb{R}^3 \Leftrightarrow A\vec{x} = \vec{b}$ has a sol
for every $\vec{b} \in \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & * \\ 2 & 5 & 7 & * \\ 0 & 2 & 5 & * \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & * \\ 0 & 1 & 3 & * \\ 0 & 2 & 5 & * \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & * \\ 0 & 1 & 3 & * \\ 0 & 0 & -1 & * \end{array} \right]$$

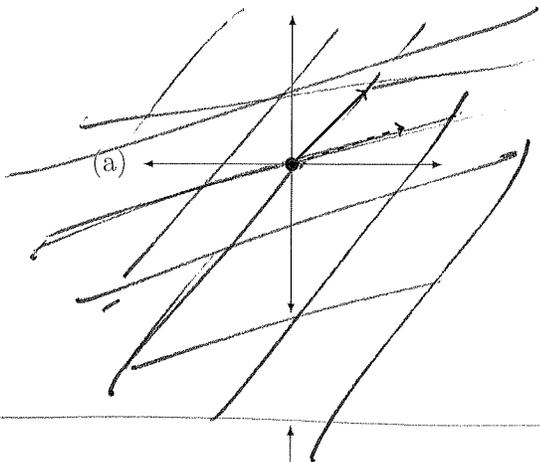
\uparrow
cannot contain $[0 \ 0 \ 0 \ | \ *]$
for any \vec{b}
 \Rightarrow always consistent

So columns of A DO span \mathbb{R}^3

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10. For each of the following vectors of \mathbb{R}^2 , sketch the lattice of their integer combinations. Then, answer the questions given.



Define $\text{Span}\{\vec{b}_1, \vec{b}_2\}$

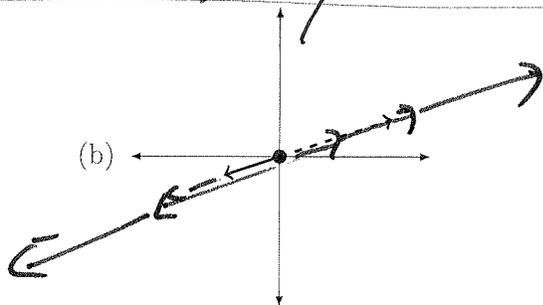
$$\{\vec{w} : \vec{w} = c_1\vec{b}_1 + c_2\vec{b}_2 \text{ for } c_1, c_2 \in \mathbb{R}\}$$

Is the span a point, line, or a plane?

a plane

Do the vectors span \mathbb{R}^2 ?

yes



Define $\text{Span}\{\vec{b}_1, \vec{b}_2\}$

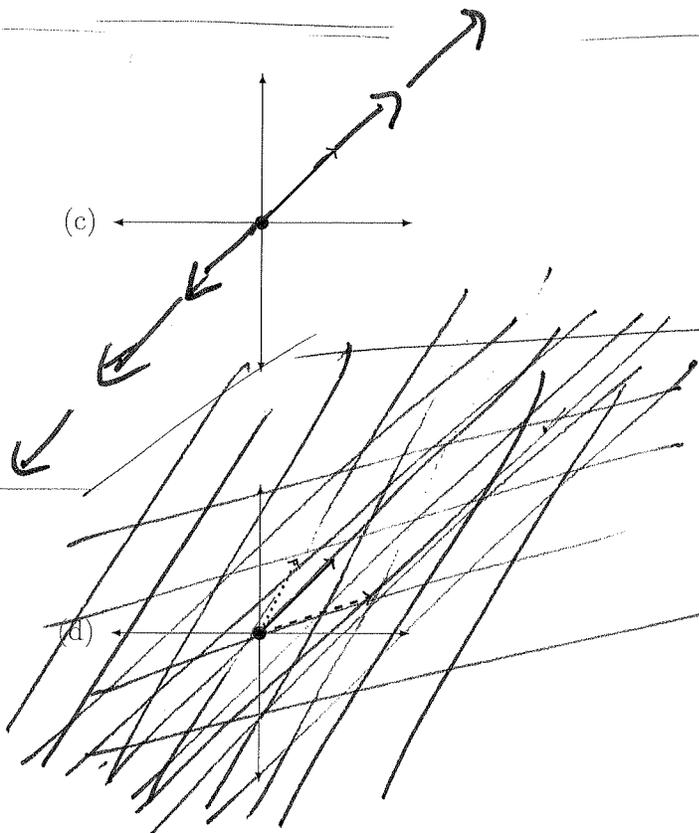
$$\{\vec{w} : \vec{w} = c_1\vec{b}_1 + c_2\vec{b}_2 \text{ for } c_1, c_2 \in \mathbb{R}\}$$

Is the span a point, line, or a plane?

a line

Do the vectors span \mathbb{R}^2 ?

No.



Define $\text{Span}\{\vec{b}_1\}$

$$\{\vec{w} : \vec{w} = c_1\vec{b}_1 \text{ for } c_1 \in \mathbb{R}\}$$

Is the span a point, line, or a plane?

a line

Do the vectors span \mathbb{R}^2 ?

No



Define $\text{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

$$\{\vec{w} : \vec{w} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 \text{ for } c_1, c_2, c_3 \in \mathbb{R}\}$$

Is the span a point, line, or a plane?

a plane

Do the vectors span \mathbb{R}^2 ?

yes.

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11. State the following definitions, and fill in the appropriate blanks in the theorems.

Definitions

(a) Give the formal definition of $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\{\vec{w} \in \mathbb{R}^m : \vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \text{ for } c_1, c_2, c_3 \in \mathbb{R}\}$$

(b) Give the geometric meaning of $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

it is the space of vectors gen'd by $\vec{v}_1, \vec{v}_2, \vec{v}_3$

(c) Given an $m \times n$ matrix A and a vector $\vec{x} \in \mathbb{R}^n$. State the definition of $A\vec{x}$.

$$[\vec{a}_1 \dots \vec{a}_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + \dots + x_n \vec{a}_n$$

(d) Suppose that $A = [\vec{a}_1 \dots \vec{a}_n]$ is an $m \times n$ matrix. Rephrase the sentence "The columns of A span \mathbb{R}^m " as a statement about vectors.

the columns of A span \mathbb{R}^m

$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}$ has a soln for every $\vec{b} \in \mathbb{R}^m$

(e) Suppose that A is an 5×7 matrix, and that $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} A \end{bmatrix}_{m \times n} \begin{bmatrix} \vec{x} \end{bmatrix}_{n \times 1} = \begin{bmatrix} \vec{b} \end{bmatrix}_{m \times 1}$$

Find j and k so that $\vec{x} \in \mathbb{R}^j$ and $\vec{b} \in \mathbb{R}^k$.

$j = \# \text{ rows in } \vec{x} = \# \text{ columns in } A = 7 \text{ . i.e. } \vec{x} \in \mathbb{R}^7 \text{ .}$

$k = \# \text{ rows in } \vec{b} = \# \text{ rows in } A = 5 \text{ . i.e. } \vec{b} \in \mathbb{R}^5 \text{ .}$

Theorems

Theorem 2 The reduced echelon form of a linear system has three possible cases

- (a) The system has \emptyset solutions if contains $(0 \dots 0 | \square)$
- (b) The system has 1 solutions if is consistent AND has pivot in ea. column
- (c) The system has ∞ -many solutions if is consistent AND has column w/o a pivot

Theorem 4: The columns of an $m \times n$ matrix A span \mathbb{R}^m

if and only if there is a pivot in every Row.